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On Fuzzy T_0 and R_0 Topological Spaces

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1. INTRODUCTION

Fuzzy T_0 spaces have been defined and studied by Hutton and Reilly [2] and Pu and Liu [3] earlier, while Hutton and Reilly [2] have also introduced fuzzy R_0 spaces. Fuzzy T_1 spaces, on the other hand, have been studied by several authors and also by us [6, 7] via a slightly different definition. This note has grown out of a desire to have a certain compatibility among the concepts of fuzzy T_0, R_0, and T_1 spaces paralleling their topological counterparts and also to equip these with some other desirable features. This has led to a modification of the existing definitions of fuzzy T_0 and R_0 concepts and our observations indicate that the new definitions are the more appropriate ones.

All undefined fuzzy topological concepts and notations used here are fairly standard by now and can be found, e.g., in Lowen [5]. Note, however, that Chang [1] and Lowen [5] have defined fuzzy topology differently and in this note we consider fuzzy topologies in the sense of Chang as well as in the sense of Lowen. Recall that a *fuzzy point* in a set X with *support* $x \in X$ and *value* $r \in (0, 1)$ is a fuzzy set in X which takes value 0 everywhere except at x, where it takes value r; we shall denote it by x_r. The s-valued constant fuzzy set in X will be denoted by s. If A is a subset of X, we shall use A to denote its characteristic function also.

2. FUZZY T_0 TOPOLOGICAL SPACES

DEFINITION 2.1. (Pu and Liu [3]). An fts (X, τ) is said to be a fuzzy T_0 topological space iff (X, τ) is quasi T_0[1] and for any $r, s \in [0, 1)$ and $x, y \in X$, $x \neq y$, there exists $U \in \tau$ such that either $U(x) = r$ and $U(y) > s$, or $U(x) > r$ and $U(y) = s$.

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[1] (X, τ) is quasi T_0 iff $\forall x \in X$ and $\rho \in [0, 1]$, $\exists B \in \tau$ with $B(x) = \rho$.

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DEFINITION 2.2. (Hutton and Reilly [2]). An fts (X, τ) is said to be fuzzy T_0 iff each fuzzy set in X can be written as $\sup_i \inf_j U_{ij}$, where U_{ij}, $i \in I$, $j \in J$, is fuzzy open or fuzzy closed.

DEFINITION 2.3. An fts (X, τ) is said to be a fuzzy T_1 topological space iff $\forall x, y \in X$, $\exists U, V \in \tau$ such that $U(x) = 1$, $U(y) = 0$, $V(y) = 1$, and $V(x) = 0$.

We point out here that fuzzy T_1 topological spaces have been defined by several authors including Hutton and Reilly [2] and Pu and Liu [3]. We shall, however, use the above definition 2.3 of fuzzy T_1-ness for reasons given in [6] and [7].

We find that a fuzzy T_1 topological space is not necessarily fuzzy T_0 in the sense of Definition 2.1 or 2.2 (see the counterexample 2.4 below). We therefore suggest another definition of a fuzzy T_0 space as follows:

DEFINITION 2.4. An fts (X, τ) is said to be fuzzy T_0 iff $\forall x, y \in X$, $x \neq y$, $\exists U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$ or $U(y) = 1$ and $U(x) = 0$.

We now compare our definition of fuzzy T_0-ness with the previous two definitions in the following theorem.

THEOREM 2.1. *Consider the following statements for an fts*(X, τ):

(I) $\forall x, y \in X$, $x \neq y$, $\exists U \in \tau$ *such that either* (a) $U(x) = 1$, $U(y) = 0$ *or* (b) $U(y) = 1$ *and* $U(x) = 0$.

(II) *Each fuzzy set in* X *can be written in the form* $\sup_i \inf_j U_{ij}$, *where each* U_{ij}, $i \in I$, $j \in J$ *is a fuzzy open or a fuzzy closed set.*

(III) (X, τ) *is quasi* T_0 *and, for any two distinct points* $x, y \in X$ *and for all* $r, s \in [0, 1)$, *there exists* $U \in \tau$ *such that either* (a) $U(x) = r$ *and* $U(y) > s$ *or* (b) $U(x) > r$ *and* $U(y) = s$.

We then have the following implications:

(I) $\not\Rightarrow$ (II) (if τ is in Chang's sense)

(I) $\Rightarrow$ (II) (if τ is in Lowen's sense)

(II) $\not\Rightarrow$ (I) (whether τ is in Chang's sense or in Lowen's sense)

(I) $\not\Rightarrow$ (III) (if τ is in Chang's sense)

(I) $\Rightarrow$ (III) (if τ is in Lowen's sense)

(III) $\not\Rightarrow$ (I) (whether τ is in Chang's sense or in Lowen's sense).

Proof. (I) $\not\Rightarrow$)(II) (if τ is in Chang's sense).

COUNTER EXAMPLE 2.1. Let X be any set. Let d be the fuzzy topology on X (in Chang's sense) consisting of all the crisp fuzzy sets in X. Then (X, d) satisfies (I) but (II) is not satisfied here since the fuzzy point x_r cannot be represented in the form $\sup_i \inf_j U_{ij}$ as all fuzzy open and fuzzy closed sets take values either 0 or 1.

(I) $\Rightarrow$ (II) (if τ is in Lowen's sense). Since each fuzzy set is the supremum of its fuzzy points and, further, since each fuzzy point x_r can be written as $\{x\} \cap \mathbf{r}$, $\mathbf{r}$ being the r-valued constant function, it is sufficient to prove that assuming (I), $\{x\}$ can be written in the form $\inf_i U_i$, where U_i is fuzzy open or fuzzy closed $\forall i$.

Now let us fix x and consider $y \in X$, $y \neq x$. Using (I), $\exists U \in \tau$ such that $U(x)=1$, $U(y)=0$ or $U(y)=1$, $U(x)=0$. Let us write $X - \{x\} = X_1 \cup X_2$, where $X_1 = \{ y_1 \in X - \{x\}: \exists U_{y_1} \in \tau$ such that $U_{y_1}(x)=1$, $U_{y_1}(y_1)=0\}$ and $X_2 = \{ y_2 \in X - \{x\}: \exists U_{y_2} \in \tau$ such that $U_{y_2}(x)=0$ and $U_{y_2}(y_2)=1\}$. Let $\mathscr{U}_1$ denote the family of fuzzy open sets $\{U_{y_1}: y_1 \in X_1\}$ and $\mathscr{U}_2$ denote the family of fuzzy closed sets $\{U'_{y_2}: y_2 \in X_2\}$. Then $\{x\} = \inf_{V \in \mathscr{U}_1 \cup \mathscr{U}_2} V$. Thus $\{x\}$ has been represented as a supremum of fuzzy sets which are fuzzy open or fuzzy closed.

(II) $\not\Rightarrow$ (I) (whether τ is in Chang's sense or in Lowen's sense).

COUNTER EXAMPLE 2.2. Let X be any nonempty set and τ be the fuzzy topology generated by $\{x'_r: x_r$ is a fuzzy point in $X\} \cup \{$all constant functions from X to $I\}$. Here each fuzzy point is fuzzy closed in X and hence it is fuzzy T_0 in the sense of Hutton. Thus the statement (II) of this theorem is satisfied here but not (I) since no fuzzy open set, except $\varnothing$, takes the value zero at any point of X.

(I) $\not\Rightarrow$ (III) (if τ is in Chang's sense). Consider the counterexample 2.1 again. Then (X, d) satisfies (I) but not (III) since it is not quasi T_0, as there is no fuzzy open set taking a value in $(0, 1)$.

(I) $\Rightarrow$ (III) (if τ is in Lowen's sense). If τ is in Lowen's sense then clearly it is quasi T_0. Further choose any $r, s \in [0, 1)$ and $x, y \in X$, $x \neq y$; then using (I), $\exists U$ such that either $U(x)=1$, $U(y)=0$ or $U(r)=0$, $U(y)=1$. Now consider $V = U \cup \mathbf{r}$. Then $V(x)=1$, $V(y)=r$ or $V(y)=1$, $V(x)=r$, implying the existence of a fuzzy open set V satisfying $V(x)>s$, $V(y)=r$ or $V(y)>s$, $V(x)=r$.

(III) $\not\Rightarrow$ (I) (whether τ is in Chang's sense or in Lowen's sense). See the counterexample 2.2. Here (X, τ) satisfies (III); for if we take any $r, s \in [0, 1)$ and $x, y \in X$, $x \neq y$, then $U = x'_{1-r} \in \tau$ and it satisfies the condition $U(x)=r$ and $U(y)>s$ but (X, τ) does not satisfy (I).

Remark 2.1. From the above theorem we observe that our definition of fuzzy T_0-ness is independent of those due to Ming and Ming and Hutton and Reilly if we consider fuzzy topology in the sense of Chang.

Both Hutton and Reilly [2] and Pu and Liu [3] have shown that their fuzzy T_0 spaces are closed under forming products. In fact, Ming and Ming have shown that using Definition 2.1, the product of fuzzy T_0 spaces is fuzzy T_0 but the converse is not true, in general. Hutton and Reilly, on the other hand, have proved that a product is fuzzy T_0 iff each factor is fuzzy T_0.

We now prove here the following theorem in which we use fuzzy T_0-ness in the sense of Definition 2.4.

THEOREM 2.2. *Let $\{(X_i, \tau_i): i \in I\}$ be a family of fuzzy T_0 spaces. Then the product $(X, \tau) = \prod_i (X_i, \tau_i)$ is fuzzy T_0 iff each coordinate fts is fuzzy T_0.*

Proof. ($\Rightarrow$) Let $x = \langle x_i \rangle$ and $y = \langle y_i \rangle \in X$, $x \neq y$. Then $x_i \neq y_i$ for at least one $i \in I$. Now since (X_i, τ_i) is fuzzy T_0, $\exists U_i \in \tau_i$ such that $U_i(x_i) = 1$, $U_i(y_i) = 0$ or $U_i(x_i) = 0$, $U_i(y_i) = 1$. Let us suppose that U_i is such that $U_i(x_i) = 1$ and $U_i(y_i) = 0$. Now consider $\prod_{j \in I} U'_j$, where $U'_j = X_j$ for $j \neq i$ and $U'_i = U_i$. Then $\prod_{j \in I} U'_j(x) = 1$ and $\prod_{j \in I} U'_j(y) = 0$. Similarly, the case when U_i satisfies $U_i(y_i) = 1$, $U_i(x_i) = 0$ can be considered. Hence the product is fuzzy T_0. Conversely, let the product be fuzzy T_0. Consider some (X_i, τ_i). Let $x_i, y_i \in X_i$, $x_i \neq y_i$. Now consider two distinct points $x = \langle x'_j \rangle$ and $y = \langle y'_j \rangle$ in X, where $x'_j = y'_j$ for $j \neq i$ and $x'_i = x_i$, $y'_i = y_i$. Then $\exists U \in \tau$ such that either $U(x) = 1$, $U(y) = 0$ or $U(x) = 0$, $U(y) = 1$. Let us assume that $\exists U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$. Take a fuzzy point x_t in U. Then we can find a basic fuzzy open set $\prod_j U^t_j$ such that $x_t \in \prod_j U^t_j \subseteq U$. On fixing x and varying t such that $0 < t < 1$ we see that

$$t < \prod_j U^t_j(x) \leqslant U(x) \qquad (\forall t, 0 < t < 1). \tag{1}$$

Clearly, $\sup_{0 < t < 1} \prod_j U^t_j(x) = 1$. Now since $U^t_j(x'_j) \leqslant \sup_t U^t_j(x'_j)$ $\forall t$, we have $\inf_j U^t_j(x'_j) \leqslant \inf_j \sup_t U^t_j(x'_j)$, $\forall t$ and hence $\sup_t \inf_j U^t_j(x'_j) \leqslant \inf_j \sup_t U^t_j(x'_j)$ or that $\sup_t \prod_j U^t_j(x) \leqslant \prod_j \sup_t U^t_j(x)$ implying that $\prod_j \sup_t U^t_j(x) = 1$. Now $\prod_j \sup_t U^t_j(x) = \inf_j \{\sup_t U^t_j(x'_j)\} = 1$ implies that $\sup_t U^t_j(x'_j) = 1$ $\forall j \in I$. In particular, $\sup_t U^t_i(x'_i) = \sup_t U^t_i(x_i) = 1$. Further since $U(y) = 0$, $\prod_j U^t_j(y) = 0$, $\forall t$, $0 < t < 1$, i.e., $\inf U^t_j(y'_j) = 0$, $\forall t$. But for $j \neq i$, $U^t_j(y'_j) = U^t_j(x'_j) > 0$ (using (1)) and hence the only possibility is that $U^t_i(y'_i) = U^t_i(y_i) = 0$, $\forall t$, $0 < t < 1$. Therefore $\sup_t U^t_i(y_i) = 0$. Thus we have a fuzzy open set $\sup_t U^t_i$ in X_i taking value 1 at x_i and zero at y_i. The case when $U(y) = 1$, $U(x) = 0$ can also be handled similarly. Thus (X_i, τ_i) is fuzzy T_0.

Let (X, τ) be an fts and (A, τ_A) a fuzzy subspace of this fts. Since a point in A is also a point in X and a fuzzy open (fuzzy closed) set in A can be written as U/A, U being fuzzy open (fuzzy closed) in X, the following theorem easily follows where fuzzy T_0-ness is again in the sense of Definition 2.4.

THEOREM 2.3. *A fuzzy subspace of a fuzzy T_0 topological space is also fuzzy T_0.*

We now prove that the three concepts of fuzzy T_0-ness arising from Definitions 2.1, 2.2, and 2.4 are good extensions of the concepts of (topological) T_0-ness in the sense of Lowen [5].

THEOREM 2.4. *Let (X, T) be a topological space. Then (X, T) is* $T_0 \Leftrightarrow (X, \omega(T))$ is fuzzy T_0 whether we use Definition 2.1, 2.2, or 2.4.

Proof. Case I. When fuzzy T_0-ness is in the sense of Definition 2.1, let (X, T) be T_0. Now $(X, \omega(T))$ is clearly quasi T_0. Further, choose, $r, s \in [0, 1)$ and $x, y \in X$, $x \neq y$, then since (X, T) is T_0, $\exists U \in T$ such that $U(x) = 1$, $U(y) = 0$ or $U(x) = 0$ and $U(y) = 1$. Now consider $U \cup \mathbf{r} = V$ (say), where U is considered as a fuzzy set. Then $V(x) = 1$, $V(y) = r$ or $V(x) = r$, $V(y) = 1$. Clearly V satisfies the conditions that $V(x) > s$, $V(y) = r$ or $V(x) = r$, $V(y) > s$ implying that $(X, \omega(T))$ is fuzzy T_0.

Conversely, let $(X, \omega(T))$ be fuzzy T_0. Let $x, y \in X$, $x \neq y$ and choose $r = s \in [0, 1)$. Then using fuzzy T_0-ness of $(X, \omega(T))$, $\exists U \in \omega(T)$ such that $U(x) > r$, $U(y) = r$ or $U(x) = r$, $U(y) > r$. Now $U^{-1}(r, 1] \in T$ and we have $x \in U^{-1}(r, 1]$, $y \notin U^{-1}(r, 1]$ or $x \notin U^{-1}(r, 1]$ and $y \in U^{-1}(r, 1]$, showing that (X, T) is T_0.

Case II. When fuzzy T_0-ness is in the sense of Definition 2.2. Let us assume that (X, T) is T_0. Then each $\{x\}$, $x \in X$, can be written in the form $\bigcap_{j \in I} U_j$, U_j open or closed. Regarding these U_j as fuzzy open or fuzzy closed sets of $(X, \omega(T))$, it follows that $\forall x \in X$, $\{x\} = \bigcap_{j \in I} U_j$, where U_j is fuzzy open or fuzzy closed in $(X, \omega(T))$. Now any fuzzy point x in X can be written as $\mathbf{r} \cap \{x\}$, $\mathbf{r}$ being the r-valued constant fuzzy set, therefore x_r can be represented as an intersection of fuzzy open or fuzzy closed sets in $(X, \omega(T))$ and hence each fuzzy set in X, being the supremum of its fuzzy points, can be written as $\sup_i \inf_j U_{ij}$, U_{ij} fuzzy open or fuzzy closed. Thus $(X, \omega(T))$ is fuzzy T_0. Conversely, let $(X, \omega(T))$ be fuzzy T_0. Let x, y be two distinct points in X. Consider the fuzzy points x_r and y_s in X. In view of the assumption that $(X, \omega(T))$ is fuzzy T_0, we can write $x_r = \sup_i \inf_j U_{ij}$, U_{ij} fuzzy open or fuzzy closed. Since $x_r(y) = 0$, $\sup_i \inf_j U_{ij}(y) = 0$ which implies that $\inf_j U_{ij}(y) = 0$, $\forall i$. Now if $0 < r' < r$ then $x_{r'} \in \sup_i \inf_j U_{ij}$ which implies that $x_{r'} \in \inf_j U_{ij}$ for some i say i_1. Then $x_{r'} \in U_{ij}$, $\forall j$ and $i = i_1$. Now since $\inf U_{ij}(y) = 0$ for $i = i_1$ and $\forall j$, by the definition of an infimum, we can find a j, say j_1, such that $U_{i_1 j_1}(y) < r'$. Now consider the fuzzy set $U_{i_1 j_1}$ in $(X, \omega(T))$. If $U_{i_1 j_1}$ is fuzzy open then $U_{i_1 j_1}^{-1}(r', 1]$ is an open set in X satisfying $x \in U_{i_1j_1}^{-1}(r', 1]$ and $y \notin U_{i_1j_1}^{-1}(r', 1]$. If $U_{i_1j_1}$ is fuzzy closed then the complement $U'_{i_1j_1}$ of U_{ij} is fuzzy open in X and $U'^{-1}_{i_1j_1}(1-r', 1]$ is an open set in X satisfying $y \in U'^{-1}_{i_1j_1}(1-r', 1]$ and $x \notin U'^{-1}_{i_1j_1}(1-r', 1]$.

Case III. When fuzzy T_0-ness is in the sense of Definition 2.4. First, let (X, T) be T_0 and let $x, y \in X$, $x \neq y$. Since (X, T) is T_0, $\exists U \in \tau$ such that either $x \in U$ and $y \notin U$ or $x \notin U$, and $y \in U$. Now U, regarded as a member of $\omega(T)$, is such that either $U(x) = 1$ and $U(y) = 0$ or $U(y) = 1$ and $U(x) = 0$, showing that $(X, \omega(T))$ is fuzzy T_0. Conversely, let $(X, \omega(T))$ be fuzzy T_0. Let $x, y \in X$, $x \neq y$. Then $\exists U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$ or $U(y) = 1$ and $U(x) = 0$. Then clearly the open set $U^{-1}(0, 1]$ in X is such that either $x \in U^{-1}(0, 1]$, $y \notin U^{-1}(0, 1]$ or $x \notin U^{-1}(0, 1]$, $y \in U^{-1}(0, 1]$. Hence (X, T) is T_0.

3. Fuzzy R_0 Spaces

Fuzzy R_0 spaces have been defined by Hutton and Reilly [2] as follows:

Definition 3.1. An fts (X, τ) is said to be fuzzy R_0 iff each fuzzy open set can be written as a supremum of fuzzy closed sets.

We have found that our fuzzy T_1 spaces are not necessarily fuzzy R_0 in this sense as shown in the following counterexample.

Counter Example 3.1. Let $X = \{x, y\}$ and τ be the fuzzy topology on X generated by $\{X, \{x\}, \{y\}, x_r$ such that $r \neq \frac{1}{2}\}$. Then (X, τ) is fuzzy T_1 as $\{x\}$ and $\{y\}$are fuzzy closed in X but it is not fuzzy R_0 since the fuzzy open set x_r cannot be expressed as a supremum of fuzzy closed sets.

Further, we find that (see Counterexample 3.4) a fuzzy R_0 topological space as defined by Hutton and Reilly [2] is not a good extension of the concept of an R_0 topological space. Therefore we propose another definition of a fuzzy R_0 space as follows:

Definition 3.2. An fts(X, τ) is fuzzy R_0 iff $\forall x, y \in X$, $x \neq y$, whenever there is a $U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(y) = 1$ and $V(x) = 0$.

The two definitions of fuzzy R_0 spaces mentioned here are totally independent as shown by the following counterexamples.

Counterexample 3.2. Let $X = \{x, y\}$ and let τ be the fuzzy topology on X generated by $\{x_{1/3}\} \cup \{$all constant maps from X to $[0, 1]\}$. Then (X, τ) is fuzzy R_0 in the sense of Definition 3.2 but it is not fuzzy R_0 in the sense of Definition 3.1, since the fuzzy open set $x_{1/3}$ cannot be expressed as a supremum of fuzzy closed sets in X.

COUNTEREXAMPLE 3.3. Let $X = \{x, y\}$. Suppose τ is the fuzzy topology on X generated by $\mathscr{S} = \{x\} \cup \{x'_r : 0 < r < 1\} \cup \{y'_r : 0 < r < 1\} \cup \{$all constant functions from X to $[0, 1]\}$. It is clear that no fuzzy open set of τ takes a zero value at x while $\{x\}$ is a fuzzy open set with value 1 at x and value 0 at y. Hence (X, τ) is not fuzzy R_0 in the sense of Definition 3.2. But (X, τ) is fuzzy R_0 in the sense of Definition 3.1, since each fuzzy point is fuzzy closed and hence each fuzzy open set can be written as a supremum of fuzzy closed sets.

Hutton and Reilly [2] have already noted closure under forming products of their fuzzy R_0 spaces and a similar result for fuzzy R_0 spaces in the sense of Definition 3.2 can be proved by a suitable adaption of the proof of the earlier theorem 2.2.

Similarly, as for fuzzy T_0 spaces, we can show that subspaces of fuzzy R_0 spaces are also fuzzy R_0 whether we use Definition 3.1 or Definition 3.2.

We next show by means of an example that the concept of a fuzzy R_0 space, as defined by Hutton and Reilly (definition 3.1) is not a good extension (in the sense of Lowen) of its topological counterpart.

COUNTEREXAMPLE 3.4. Let $X = \{x, y, z\}$ and $T = \{\varnothing, X, \{x\}, \{y\}, \{x, y\}\}$. T is then easily seen to be an R_0 topology on X. Consider now $\omega(T)$ and the fuzzy open set $\{x, y\} \in \omega(T)$. We claim that $\{x, y\}$ cannot be expressed as a supremum of fuzzy closed sets of $(X, \omega(T))$ for if it were, then $\{x, y\} = \sup_i F_i$, F_i being fuzzy closed. But $\sup_i F_i = \sup_i (X - U_i) = X - \bigcap_i U_i$, where U_i are fuzzy open. So $\{x, y\} = \sup_i F_i$ iff $\bigcap_i U_i = \{z\}$ which implies that $U_i(z) = 1$ $\forall i$. Recalling that members of $\omega(T)$ are l.s.c. functions from X to $[0, 1]$, we see that if U_i: $(X, T) \to [0, 1]$ with $U_i(z) = 1$ is l.s.c. then $U_i^{-1}(a, 1] = X$, $\forall a \in [0, 1)$ and so $U_i = X$, $\forall i$. Thus, writing $\{x, y\} = X - \bigcap_i U_i = \sup_i F_i$, is impossible. This shows that $(X, \omega(T))$ cannot be fuzzy R_0 in the sense of Hutton and Reilly [2].

Our next observation shows that our Definition 3.2 of a fuzzy R_0 space is better behaved in this respect.

THEOREM 3.1. *A topological space* (X, T) *is* R_0 *iff the fts* $(X, \omega(T))$ *is fuzzy* R_0 (*in the sense of Definition* 3.2).

Proof. Let us first assume that (X, T) is R_0. Let $x, y \in X$, $x \neq y$. Suppose $\exists U \in \omega(T)$ such that $U(x) = 1$, $U(y) = 0$. Then $U^{-1}(0, 1] \in T$ and is such that $x \in U^{-1}(0, 1]$, $y \notin U^{-1}(0, 1]$; therefore since (X, T) is R_0, $\exists V \in T$ such that $x \notin V$, $y \in V$. Now considering V as a member of $\omega(T)$, we see that

$V(y)=1$, $V(x)=0$. Hence $(X, \omega(T))$ is fuzzy R_0. Conversely, let $(X, \omega(T))$ be fuzzy R_0 and let $x, y \in X$, $x \neq y$. Suppose that $\exists U \in T$ such that $x \in U$, $y \notin U$; then the crisp fuzzy open set U is such that $U(x)=1$, $U(y)=0$ and so, using the fuzzy R_0 property of $(X, \omega(T))$, $\exists V \in \omega(T)$ such that $V(x)=0$, $V(y)=1$. Now $V^{-1}(0,1] \in T$ and is such that $x \notin V^{-1}(0,1]$ and $y \in V^{-1}(0,1]$. This shows that (X, T) is R_0.

We close by suggesting that it appears more appropriate to define fuzzy T_0 via Definition 2.4 and fuzzy R_0 via Definition 3.2. The chief reason is that these concepts retain all the properties of the similar concepts defined by other authors but blend well with the concept of fuzzy T_1-ness defined via Definition 2.3 in the sense that fuzzy T_1-ness = fuzzy T_0-ness + fuzzy R_0-ness.

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